On the Color-Singlet States in Many-Quark Model with the *su*(4)-Algebraic Structure. I

—— *Color-Symmetric Form* ——

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In the modified Bonn quark model in which quark-pairing and particle-hole interactions are included, color-singlet states are examined in addition to the color-neutral quark-triplet state by a method of boson realization. The region which consists of single-quarks, quark pairs and quark triplets is analyzed.

§1. Introduction

It is believed from a theoretical viewpoint that an attractive quark-quark interaction leads to a quark pairing state, namely, a color superconducting state may exist in a quark matter at high density.[1)] As an effective model including the quark pairing interaction, the Bonn quark model is an interesting one.[2)] This model has originally been introduced in order to describe a nucleus as a system of interacting quarks. This model actually has the interesting feature of the formation of the color-neutral quark-triplet as a nucleon and a Δ-particle.

Recently, the Bonn quark model has been reinevestigated by the present authors in Ref.3), which is hereafter referred to as (A), in order to study possible states in the many-quark system. It was there indicated that the original Bonn quark model has an $su(4)$-dynamical symmetry. Thus, preserving only the color $su(3)$-symmetry, an $su(4)$-symmetry breaking term is introduced, which represents a particle-hole type interaction in terms of the quark shell model. We call this model the modified Bonn quark model. In (A), exact eigenstates were derived by using of the boson realization technique. Further, in Ref.4), which is hereafter referred to as (B), exact eigenstates with single-quark, quark-pairing and quark-triplet states are treated in a unified way. Also, in Ref.5), which is referred to as (C), the ground state was investigated in the modified Bonn quark model as a function of the quark number and the particle-hole interaction strength owing to the $su(4)$-symmetry breaking. It was shown that the quark-pairing state or the color-neutral quark-triplet state was realized in a certain parameter region including the transition region of both states.

In this paper, we examine a "color-singlet" state in the modified Bonn quark model. In (A), (B) and (C), the color-neutral quark-triplet state was only realized as the color-singlet state. The purpose of this paper is to reinvestigate the color-singlet

state. We will give a certain condition for a state to be a "color-singlet" in the boson realization. It will be shown that this condition leads to the superposition of the eigenstates constructed on minimum weight states in the $su(3)$-subalgebras which are included in the $su(4)$-algebra governing the original Bonn quark model. And, we must point out that color-symmetric form of a modified Bonn quark model enables us to treat the present model in terms of the above-mentioned method. The form adopted in (A) has been not based on the color-symmetric form.

This paper is organized as follows: In the next section, an outline of the modified Bonn quark model is given and a concept of "color-singlet" state is explained. In §3, the method of the boson realization in this model is given in a slightly different manner from that developed in (A), that is, the color-symmetric form. In §4, energy eigenstates are constructed on a single minimum weight state of $su(3)$-subalgebra. In §5, the energy eigenstates are extended and are constructed as a superposition based on the minimum weight states of possible $su(3)$-subalgebra. In §6, the "color-singlet" state minimizing the eigenvalue of the $su(3)$-Casimir operator is examined and the regions which consist of single-quarks, quark-pairs and quark-triplets are depicted in a parameter space. The last section is devoted to a brief summary.

§2. Outline of the model

The model treated in this paper has been already discussed in our three papers, (A), (B) and (C), in which, concerning the color-singlet states, we examined only the quark-triplet states. In this section, we will give outline of the model, including certain new aspects to be investigated. Of course, the new aspects will play a central role for the understanding of the color-singlet states in general cases. This model is formulated in terms of the $su(4)$-generators constructed by the bilinear forms of the quark creation and annihilation operators. The color quantum numbers are specified as $i = 1$, 2 and 3. Each color state has the degeneracy 2Ω. Here, $2\Omega = 2j_s + 1$ and j_s is a half-integer. Any single-particle state is specified as (i, m) with $m = -j_s, \ -j_s+1, \cdots, \ j_s - 1$ and j_s, and the quark creation and annihilation operators in (i, m) are denoted by c_{im}^* and c_{im}, respectively. For simplicity, we neglect the degrees of freedom related to the isospin. We define the following bilinear forms for the quark creation and annihilation operators:

$$
\begin{aligned}
&\widetilde{S}^1 = \sum_m c_{2m}^* c_{3\widetilde{m}}^* \ , \qquad \widetilde{S}^2 = \sum_m c_{3m}^* c_{1\widetilde{m}}^* \ , \qquad \widetilde{S}^3 = \sum_m c_{1m}^* c_{2\widetilde{m}}^* \ , \\
&\widetilde{S}_1 = (\widetilde{S}^1)^* \ , \qquad \widetilde{S}_2 = (\widetilde{S}^2)^* \ , \qquad \widetilde{S}_3 = (\widetilde{S}^3)^* \ , \qquad\qquad (2\cdot 1\mathrm{a}) \\
&\widetilde{S}_1^2 = \sum_m c_{2m}^* c_{1m} \ , \qquad \widetilde{S}_2^3 = \sum_m c_{3m}^* c_{2m} \ , \qquad \widetilde{S}_3^1 = \sum_m c_{1m}^* c_{3m} \ , \\
&\widetilde{S}_2^1 = (\widetilde{S}_1^2)^* \ , \qquad \widetilde{S}_3^2 = (\widetilde{S}_2^3)^* \ , \qquad \widetilde{S}_1^3 = (\widetilde{S}_3^1)^* \ , \qquad\qquad (2\cdot 1\mathrm{b}) \\
&\widetilde{S}_1^1 = \sum_m (c_{2m}^* c_{2m} + c_{3m}^* c_{3m}) - 2\Omega \ , \qquad \widetilde{S}_2^2 = \sum_m (c_{3m}^* c_{3m} + c_{1m}^* c_{1m}) - 2\Omega \ , \\
&\widetilde{S}_3^3 = \sum_m (c_{1m}^* c_{1m} + c_{2m}^* c_{2m}) - 2\Omega \ . \qquad\qquad (2\cdot 1\mathrm{c})
\end{aligned}
$$

Here, $c^*_{i\widetilde{m}} = (-)^{j_s - m} c^*_{i-m}$. The form (2·1c) gives us

$$\widetilde{N}_i = \widetilde{N} - \widetilde{S}^i_i - 2\Omega \ , \qquad \widetilde{N} = \sum_i \widetilde{N}_i \ . \tag{2·1d}$$

Here, $\widetilde{N}_i$ and $\widetilde{N}$ denote the quark number operator in the color i and the total quark number operator, respectively. The operators defined in the relation (2·1) are generators of the $su(4)$-algebra:

$$\begin{aligned}
&[\ \widetilde{S}^i\ ,\ \widetilde{S}^j\ ] = 0\ , \qquad [\ \widetilde{S}^i\ ,\ \widetilde{S}_j\ ] = \widetilde{S}^j_i\ ,\\
&[\ \widetilde{S}^j_i\ ,\ \widetilde{S}^k\ ] = \delta_{ij}\widetilde{S}^k + \delta_{jk}\widetilde{S}^i\ ,\\
&[\ \widetilde{S}^j_i\ ,\ \widetilde{S}^k_l\ ] = \delta_{jl}\widetilde{S}^k_i - \delta_{ik}\widetilde{S}^j_l\ .
\end{aligned} \tag{2·2a}$$

The Casimir operator of the $su(4)$-algebra, which we denote as $\widetilde{\boldsymbol{P}}^2$, is expressed in the form

$$\begin{aligned}
\widetilde{\boldsymbol{P}}^2 = &\sum_i \left(\widetilde{S}_i\widetilde{S}^i + \widetilde{S}^i\widetilde{S}_i\right) + \sum_{i\neq j}\widetilde{S}^i_j\widetilde{S}^j_i\\
&+\frac{1}{4}\left[\left(\widetilde{S}^2_2 - \widetilde{S}^3_3\right)^2 + \left(\widetilde{S}^3_3 - \widetilde{S}^1_1\right)^2 + \left(\widetilde{S}^1_1 - \widetilde{S}^2_2\right)^2\right]\ .
\end{aligned} \tag{2·2b}$$

As a sub-algebra, the $su(4)$-algebra contains the $su(3)$-algebra. The following six operators play a part of eight kinds of the $su(3)$-generators:

$$\widetilde{S}^2_1\ , \quad \widetilde{S}^3_2\ , \quad \widetilde{S}^1_3\ , \quad \widetilde{S}^1_2\ , \quad \widetilde{S}^2_3\ , \quad \widetilde{S}^3_1\ . \tag{2·3}$$

Concerning the choice of the remaining two, formally, there exist six cases:

$$\frac{1}{2}(\widetilde{S}^2_2 - \widetilde{S}^3_3)\ , \qquad \widetilde{S}^1_1 - \frac{1}{2}(\widetilde{S}^2_2 + \widetilde{S}^3_3)\ , \tag{2·4a}$$

$$\frac{1}{2}(\widetilde{S}^3_3 - \widetilde{S}^1_1)\ , \qquad \widetilde{S}^2_2 - \frac{1}{2}(\widetilde{S}^3_3 + \widetilde{S}^1_1)\ , \tag{2·4b}$$

$$\frac{1}{2}(\widetilde{S}^1_1 - \widetilde{S}^2_2)\ , \qquad \widetilde{S}^3_3 - \frac{1}{2}(\widetilde{S}^1_1 + \widetilde{S}^2_2)\ , \tag{2·4c}$$

$$\frac{1}{2}(\widetilde{S}^3_3 - \widetilde{S}^2_2)\ , \qquad \widetilde{S}^1_1 - \frac{1}{2}(\widetilde{S}^3_3 + \widetilde{S}^2_2)\ , \tag{2·5a}$$

$$\frac{1}{2}(\widetilde{S}^1_1 - \widetilde{S}^3_3)\ , \qquad \widetilde{S}^2_2 - \frac{1}{2}(\widetilde{S}^1_1 + \widetilde{S}^3_3)\ , \tag{2·5b}$$

$$\frac{1}{2}(\widetilde{S}^2_2 - \widetilde{S}^1_1)\ , \qquad \widetilde{S}^3_3 - \frac{1}{2}(\widetilde{S}^2_2 + \widetilde{S}^1_1)\ . \tag{2·5c}$$

By transposing $\widetilde{S}^2_2$ and $\widetilde{S}^3_3$, the case (2·5a) is obtained from the case (2·4a). The cases (2·4b) and (2·4c) and also the cases (2·5b) and (2·5c) are obtained from the cases (2·4a) and (2·5a), respectively, by the cyclic permutation $(1 \to 2\ ,\ 2 \to 3,\ 3 \to 1)$. Usually, the eight $su(3)$-generators are composed of three $su(2)$-generators and its scalar and two sets of spinors. For each case, we can express the $su(3)$-generators in

terms of $(\widetilde{S}_i^j)$. In Appendix A, an example is shown for the case (2·4a). For these six cases, the Casimir operator, which we denote as $\widetilde{\boldsymbol{Q}}^2$, can be expressed in the form

$$\widetilde{\boldsymbol{Q}}^2 = \sum_{i\neq j} \widetilde{S}_j^i \widetilde{S}_i^j + \frac{1}{3}\left[\left(\widetilde{S}_2^2 - \widetilde{S}_3^3\right)^2 + \left(\widetilde{S}_3^3 - \widetilde{S}_1^1\right)^2 + \left(\widetilde{S}_1^1 - \widetilde{S}_2^2\right)^2\right]$$

$$= \sum_{i\neq j} \widetilde{S}_j^i \widetilde{S}_i^j + \sum_i (\widetilde{S}_i^i)^2 - \frac{1}{3}\left(\sum_i \widetilde{S}_i^i\right)^2 . \qquad (2·6)$$

Naturally, the form (2·6) is common to the six cases. It should be noted that the treatment in (A) $\sim$ (C) has been restricted only to the case (2·4a). In this paper, we examine the remaining five cases on an equal footing with the case (2·4a).

The Hamiltonian of the present model is expressed in the following form:

$$\widetilde{H}_m = \widetilde{H} + \chi\widetilde{\boldsymbol{Q}}^2 , \qquad \widetilde{H} = -\sum_i \widetilde{S}^i \widetilde{S}_i . \qquad (2·7)$$

Here, χ denotes a real parameter. In the original model, which is usually called as the Bonn model, only the term $\widetilde{H}$ is taken into account. In (A) $\sim$ (C), we added new term $\chi\widetilde{\boldsymbol{Q}}^2$, which, as will be shown in §6, plays a fundamental role. The Hamiltonian $\widetilde{H}_m$ satisfies the relation

$$[\ \widetilde{H}_m\ , \widetilde{S}_j^i\ ] = 0 \quad \text{for} \quad i,\ j = 1,\ 2 \text{ and } 3\ , \qquad (2·8a)$$

$$\widetilde{H}_m = \frac{1}{2}(1+2\chi)\widetilde{\boldsymbol{Q}}^2 - \frac{1}{2}\left(\widetilde{\boldsymbol{P}}^2 - \widetilde{\varSigma}\right) ,$$

$$\widetilde{\varSigma} = \frac{1}{12}\left(\sum_i \widetilde{S}_i^i\right)\left(\sum_i \widetilde{S}_i^i - 12\right) = \frac{1}{3}(3\varOmega - \widetilde{N})(3\varOmega - \widetilde{N} + 6) . \qquad (2·8b)$$

The relation (2·8) indicates that the present model has the $su(3)$ color symmetry. It may be interesting to solve the eigenvalue equation for $\widetilde{H}_m$ and analyze the present model in terms of focusing on the $su(3)$ color-symmetry.

In relation to quantum chromodynamics (QCD), the eigenstate of $\widetilde{H}_m$ should obey the condition for the color singlet:

$$\widetilde{S}_i^j|cs) = 0 \quad \text{for} \quad i \neq j\ , \qquad (2·9a)$$

$$\widetilde{S}_1^1|cs) = \widetilde{S}_2^2|cs) = \widetilde{S}_3^3|cs)\ . \qquad (2·9b)$$

Here, $|cs)$ denotes an eigenstate of $\widetilde{H}_m$. The condition (2·9) gives us

$$\widetilde{\boldsymbol{Q}}^2|cs) = 0\ . \qquad (2·10a)$$

Further, the relations (2·1c), (2·1d) and (2·9b) lead us to

$$\widetilde{N}_1|cs) = \widetilde{N}_2|cs) = \widetilde{N}_3|cs) = \frac{N}{3}|cs)\ . \qquad (2·10b)$$

Here, N denotes the total quark number in the state $|cs)$. From the relation (2·10b), we can learn that, only in the case where N is a multiple of 3, the color-singlet state

appears. All the states except $|cs)$ are not color-singlet. Then, it may be desirable for the present model to expand the concept of the color-singlet state by requiring that the condition (2·9) is satisfied by the state $|cs)$ in the average.

Preparatory to accomplishment of the above aim, first, we consider the state $|cs)_L$ which obeys the condition

$$(cs|\widetilde{S}_i^j|cs)_L = 0 \quad \text{for} \quad i \neq j \ , \tag{2·11a}$$

$$(cs|\widetilde{S}_1^1|cs)_L = (cs|\widetilde{S}_2^2|cs)_L = (cs|\widetilde{S}_3^3|cs)_L \ . \tag{2·11b}$$

As can be seen in the relation (2·6), $\widetilde{\boldsymbol{Q}}^2$ is of the positive-definite quadratic form for the $su(3)$-generators and, then, we have

$$(cs|\widetilde{\boldsymbol{Q}}^2|cs)_L \geq 0 \ . \tag{2·12a}$$

Further, we have

$$(cs|\widetilde{N}_1|cs)_L = (cs|\widetilde{N}_2|cs)_L = (cs|\widetilde{N}_3|cs)_L = \frac{N}{3} \ . \tag{2·12b}$$

Since $|cs)_L$ is generally not the eigenstate of $\widetilde{N}_1$, $\widetilde{N}_2$ and $\widetilde{N}_3$, N is not restricted to be a multiple of 3. The case $(cs|\widetilde{\boldsymbol{Q}}^2|cs) = 0$ gives us $\widetilde{\boldsymbol{Q}}^2|cs) = 0$, which means $|cs)_L = |cs)$. It may be self-evident that $|cs)$ is a member of the set $\{|cs)_L\}$ and rather many of $|cs)_L$ are not color-singlet states $|cs)$. But, they leave the trace of the color-singlet states in the sense of the relation (2·11).

The condition (2·9) seems to be rather restrictive. Actually, it may be seldom to find effective theories of QCD, in which any state obeys the condition (2·9) exactly. The present model is also not the exception. Then, regarding the condition (2·11) as an approximation to the condition (2·9), in this paper, we will investigate the state $|cs)_L$, which, hereafter, we denote as the "color-singlet" state.

Further, in this case, we supplement the following requirement: It may be desirable to search the state $|cs)_L$, in which the expectation value $(cs|\widetilde{\boldsymbol{Q}}^2|cs)_L$ is as small as possible, because, in some sense, $(cs|\widetilde{\boldsymbol{Q}}^2|cs)_L$ plays a role of the standard deviation for the expectation values (2·11). We denote the state $|cs)_L$ obeying this requirement as $|cs)_M$. From the next section, we will search an idea for constructing the counterparts of $|cs)_L$ and $|cs)_M$ presented in the Schwinger boson representation. We denote them as $|cs\rangle_L$ and $|cs\rangle_M$, respectively.

§3. Boson realization

Instead of solving our problem directly in the original fermion space, in (A) $\sim$ (C), we adopt the idea of the boson realization. As was shown in Ref.6), there exist infinite possibilities for the boson realization of the $su(4)$-algebra. As a possible choice, we formulated a possible form of the boson realization in terms of the following form:

$$\widetilde{S}^i \to \hat{S}^i = \hat{a}_i^*\hat{b} - \hat{a}^*\hat{b}_i \ , \qquad \widetilde{S}_i \to \hat{S}_i = \hat{b}^*\hat{a}_i - \hat{b}_i^*\hat{a} \ ,$$

$$\widetilde{S}_i^j \to \hat{S}_i^j = (\hat{a}_i^*\hat{a}_j - \hat{b}_j^*\hat{b}_i) + \delta_{ij}(\hat{a}^*\hat{a} - \hat{b}^*\hat{b}) \ . \tag{3·1}$$

Here, $(\hat{a}_i, \hat{a}_i^*)$, $(\hat{b}_i, \hat{b}_i^*)$, $(\hat{a}, \hat{a}^*)$ and $(\hat{b}, \hat{b}^*)$ $(i = 1, 2, 3)$ denote the eight kinds of boson operators. In (A), we have discussed the reason why the representation (3·1) is acceptable. In relation to the representation (3·1), we can define the $su(1,1)$-algebra in the present boson space:

$$\hat{T}_+ = \sum_i \hat{b}_i^* \hat{a}_i^* + \hat{b}^* \hat{a}^* \ , \qquad \hat{T}_- = \sum_i \hat{a}_i \hat{b}_i + \hat{a}\hat{b} \ ,$$
$$\hat{T}_0 = \frac{1}{2}\sum_i (\hat{b}_i^* \hat{b}_i + \hat{a}_i^* \hat{a}_i) + \frac{1}{2}(\hat{b}^*\hat{b} + \hat{a}^*\hat{a}) + 2 \ . \tag{3·2}$$

They satisfy

$$[\ \hat{T}_+ \ , \ \hat{T}_- \ ] = -2\hat{T}_0 \ , \qquad [\ \hat{T}_0 \ , \ \hat{T}_\pm \ ] = \pm \hat{T}_\pm \ , \tag{3·3a}$$
$$[\ \text{any of } (\hat{T}_{\pm,0}) \ , \ \text{any of } (\hat{S}^i, \ \hat{S}_i, \ \hat{S}_i^j) \ ] = 0 \ . \tag{3·3b}$$

We cannot find any counterpart of $(\hat{T}_{\pm,0})$ in the original fermion space. Since, in (A) $\sim$ (C), we discussed the role of the $su(1,1)$-algebra in our model, in this paper, we will not repeat its discussion in detail.

With the use of the correspondence (3·1), the Hamiltonian in the boson space which comes from $\widetilde{H}_m$ given in the relation (2·7) is obtained:

$$\widetilde{H}_m \to \hat{H}_m \ , \qquad \widetilde{H} \to \hat{H} \ , \qquad \widetilde{\boldsymbol{Q}}^2 \to \hat{\boldsymbol{Q}}^2 \ . \tag{3·4}$$

By solving the eigenvalue equation for $\hat{H}_m$, we are able to get various informations on the boson system. However, through these informations, we do not obtain any knowledge on the original fermion system. The reason is very simple: The present boson system does not contain the quantity related to Ω, which characterizes the many-fermion model under investigation. In (A), we showed that the c-number Ω in the original fermion space is defined in terms of the q-number $\hat{\Omega}$ in the boson space:

$$\Omega \to \hat{\Omega} = n_0 + \frac{1}{2}(\hat{a}^*\hat{a} + \hat{b}^*\hat{b}) + \frac{1}{2}\sum_i (\hat{a}_i^*\hat{a}_i + \hat{b}_i^*\hat{b}_i) \ . \tag{3·5}$$

Here, n_0 is a certain c-number which plays a role similar to the seniority number in the $su(2)$-pairing model. In the original paper of the Bonn model, n_0 is treated as the number of the Δ-particles. In this paper, we make the positions of $\hat{\Omega}$ and n_0 reverse, that is, we regard $\hat{\Omega}$ and n_0 as c- and q-numbers, respectively. Then, $\hat{n}_0$ is expressed as

$$\hat{n}_0 = \Omega - \frac{1}{2}(\hat{a}^*\hat{a} + \hat{b}^*\hat{b}) - \frac{1}{2}\sum_i (\hat{a}_i^*\hat{a}_i + \hat{b}_i^*\hat{b}_i) \ . \tag{3·6}$$

In relation to $\hat{n}_0$, we introduce the operator $\hat{n}$ defined as

$$\hat{n} = \Omega + \frac{1}{2}(\hat{a}^*\hat{a} - \hat{b}^*\hat{b}) - \frac{1}{2}\sum_i (\hat{a}_i^*\hat{a}_i - \hat{b}_i^*\hat{b}_i) \ . \tag{3·7}$$

Later, we will mention the meaning of $\hat{n}$. Further, in (A), we introduced the operator $\hat{N}$ which comes from the total quark number. Through the above mentioned reversal, $\hat{N}$ can be expressed in the form

$$\hat{N} = 3\Omega + \frac{3}{2}(\hat{a}^*\hat{a} - \hat{b}^*\hat{b}) + \frac{1}{2}\sum_i(\hat{a}_i^*\hat{a}_i - \hat{b}_i^*\hat{b}_i) \ . \tag{3·8}$$

Associated with $\hat{N}$, the quark number operator $\hat{N}_i$ in the color i is expressed as

$$\hat{N}_i = \Omega + \frac{1}{2}(\hat{a}^*\hat{a} - \hat{b}^*\hat{b}) + \frac{1}{2}\sum_j(\hat{a}_j^*\hat{a}_j - \hat{b}_j^*\hat{b}_j) - (\hat{a}_i^*\hat{a}_i - \hat{b}_i^*\hat{b}_i) \ . \tag{3·9}$$

Of course, we have $\hat{N} = \sum_i \hat{N}_i$. The Casimir operator $\hat{\boldsymbol{Q}}^2$ can be expressed as

$$\hat{\boldsymbol{Q}}^2 = \sum_{ij}(\hat{a}_j^*\hat{a}_i - \hat{b}_i^*\hat{b}_j)(\hat{a}_i^*\hat{a}_j - \hat{b}_j^*\hat{b}_i) - \frac{1}{3}\left(\sum_i(\hat{a}_i^*\hat{a}_i - \hat{b}_i^*\hat{b}_i)\right)^2 \ . \tag{3·10}$$

We must stress that the operators $\hat{n}_0$, $\hat{n}$, $\hat{N}$ and $\hat{\boldsymbol{Q}}^2$ are color-symmetric and commute with $\hat{S}_i^j$.

As a possible idea for expressing the eigenstates of $\hat{H}_m$, in (B), we introduced the following operators:

$$\hat{q}^i = \hat{b}_i^*\hat{b} - \hat{a}^*\hat{a}_i \ , \qquad \hat{q}_i = \hat{b}^*\hat{b}_i - \hat{a}_i^*\hat{a} \ , \tag{3·11}$$

$$\hat{B}^* = \sum_i \hat{q}^i\hat{S}^i \ , \qquad \hat{B} = \sum_i \hat{S}_i\hat{q}_i \ . \tag{3·12}$$

The commutation relations between $(\hat{n}_0, \hat{n}, \hat{N})$ and $(\hat{q}^i, \hat{S}^i, \hat{B}^*)$ are given in the form

$$[\ \hat{n}_0\ ,\ \hat{q}^i\ ] = 0\ , \qquad [\ \hat{n}_0\ ,\ \hat{S}^i\ ] = 0\ , \qquad [\ \hat{n}_0\ ,\ \hat{B}^*\ ] = 0\ , \tag{3·13}$$

$$[\ \hat{n}\ ,\ \hat{q}^i\ ] = \hat{q}^i\ , \qquad [\ \hat{n}\ ,\ \hat{S}^i\ ] = 0\ , \qquad [\ \hat{n}\ ,\ \hat{B}^*\ ] = \hat{B}^*\ , \tag{3·14}$$

$$[\ \hat{N}\ ,\ \hat{q}^i\ ] = \hat{q}^i\ , \qquad [\ \hat{N}\ ,\ \hat{S}^i\ ] = 2\hat{S}^i\ , \qquad [\ \hat{N}\ ,\ \hat{B}^*\ ] = 3\hat{B}^*\ . \tag{3·15}$$

The relation (3·15) may be interesting. The operators $\hat{q}^i$, $\hat{S}^i$ and $\hat{B}^*$ carry one, two and three quarks, respectively. In (B), we mentioned that $\hat{q}^i$, $\hat{S}^i$ and $\hat{B}^*$ represent single-quark, quark-pair and quark-triplet, respectively. Concerning the relations to the $su(3)$-generators, we have the following commutation relations:

$$[\ \hat{S}_i^j\ ,\ \hat{S}^k\ ] = \delta_{jk}\hat{S}^i + \delta_{ij}\hat{S}^k\ , \tag{3·16}$$

$$[\ \hat{S}_i^j\ ,\ \hat{q}^k\ ] = -\delta_{ik}\hat{q}^j + \delta_{ij}\hat{q}^k\ , \tag{3·17}$$

$$[\ \hat{S}_i^j\ ,\ \hat{B}^*\ ] = 2\delta_{ij}\hat{B}^*\ . \tag{3·18}$$

Through the relation (3·18), we can see that $\hat{B}^*$ is a color-singlet operator.

The above is a reformulation of the boson realization for the present many-quark model which has been discussed in (A) $\sim$ (C). We do not know the operator which plays the same role as that of $\hat{q}^i$ in the original fermion space. Even if it is possible, the form may be too complicated to handle it easily. This is a basic merit for treating the present fermion model in the boson space.

§4. Energy eigenstates constructed on a chosen single minimum weight state

First, we will show the orthogonal set discussed in (A)$\sim$ (C). The discussion in (A) is based on the case (2·4a) in which the $su(3)$-generators are defined in the form (A·3). Then, for the minimum weight state $|m_1\rangle$, we set up the following relations:

$$\hat{S}_1|m_1\rangle = \hat{S}_2|m_1\rangle = \hat{S}_3|m_1\rangle = 0 \ , \tag{4·1a}$$
$$\hat{S}_2^1|m_1\rangle = \hat{S}_3^1|m_1\rangle = \hat{S}_3^2|m_1\rangle = 0 \ , \tag{4·1b}$$
$$\hat{S}_1^1|m_1\rangle = -2\sigma|m_1\rangle \ , \qquad \hat{S}_2^2|m_1\rangle = -2\sigma_0|m_1\rangle \ , \qquad \hat{S}_3^3|m_1\rangle = -2\sigma_0'|m_1\rangle \ . \tag{4·1c}$$

The relation (4·1c) leads us to

$$\left(\hat{S}_1^1 - \frac{1}{2}(\hat{S}_2^2+\hat{S}_3^3)\right)|m_1\rangle = -2\left(\sigma - \frac{1}{2}(\sigma_0+\sigma_0')\right)|m_1\rangle \ ,$$
$$\frac{1}{2}(\hat{S}_2^2-\hat{S}_3^3)|m_1\rangle = -(\sigma_0-\sigma_0')|m_1\rangle \ . \tag{4·1d}$$

As a relation which has no counterpart in the original fermion space, we require

$$\hat{T}_-|m_1\rangle = 0 \ . \tag{4·2}$$

Under the expressions (3·1) and (3·2), the relations (4·1) and (4·2) gives the following explicit form for $|m_1\rangle$:

$$|m_1\rangle = (\hat{b}_1^*)^{2(\sigma-\sigma_0)}(\hat{b}^*)^{2\sigma_0}|0\rangle \ . \quad (\sigma_0'=\sigma_0) \tag{4·3}$$

It is noted that the boson realization (3·1) leads to the solution $\sigma_0'=\sigma_0$ and in (A), we used the notation σ_1 for σ. The state $|m_1\rangle$ is specified by two quantum numbers σ_0 and σ. The relations (4·1b) and (4·1d) tell that $|m_1\rangle$ is also a minimum weight state of the $su(3)$-algebra and the eigenvalues shown in the relation (4·1d) are $-2(\sigma-(\sigma_0+\sigma_0')/2) = -2(\sigma-\sigma_0)$ and $-(\sigma_0-\sigma_0')=0$, respectively. As is shown in the relation (A·2a) and (A·3a), $\hat{I}_+ = \hat{S}_2^3$, $\hat{I}_- = \hat{S}_3^2$ and $\hat{I}_0 = (\hat{S}_2^2-\hat{S}_3^3)/2$ form the $su(2)$-algebra and $|m_1\rangle$ is a state with the $su(2)$-spin$=0$. By operating $\hat{S}^1$, $\hat{S}^2$, $\hat{S}^3$, $\hat{S}_1^2$, $\hat{S}_1^3$ and $\hat{S}_2^3$, together with $\hat{S}_1^1$, $\hat{S}_2^2$ and $\hat{S}_3^3$ on $|m_1\rangle$ appropriately, we obtain the eigenstates of $\hat{H}_m$. For example, we have

$$|\lambda\rho\sigma_0\sigma\rangle = (\hat{S}^3)^{2\lambda}(\hat{S}^4)^{2\rho}|m_1\rangle \ , \tag{4·4a}$$
$$\hat{S}^4 = \hat{S}^1\left(\hat{S}_1^1 - \frac{1}{2}(\hat{S}_2^2+\hat{S}_3^3)\right) + \hat{S}^2\hat{S}_1^2 + \hat{S}^3\hat{S}_1^3 \ . \tag{4·4b}$$

In (A) and (B), we showed the expression (4·4) in the relations (A·4·24) and (B·5·1) in notations slightly different from the above.

Next, we investigate the physical meaning and the role of the operators $\hat{n}_0$ and $\hat{n}$. It is easily verified that $|m_1\rangle$ is an eigenstate of $\hat{n}_0$ and $\hat{n}$:

$$\hat{n}_0|m_1\rangle = n_0|m_1\rangle \ , \quad n_0 = \Omega - \sigma \ , \quad \text{i.e.,} \quad \sigma = \Omega - n_0 \ , \tag{4·5a}$$
$$\hat{n}|m_1\rangle = n|m_1\rangle \ , \quad n = \Omega + \sigma - 2\sigma_0 \ , \quad \text{i.e.,} \quad \sigma_0 = \Omega - \frac{1}{2}(n_0+n) \ . \tag{4·5b}$$

On the other hand, we have

$$\hat{N}_1|m_1\rangle = n|m_1\rangle\ , \qquad \hat{N}_2|m_1\rangle = \hat{N}_3|m_1\rangle = n_0|m_1\rangle\ . \tag{4·6}$$

We can learn in the relation (4·6) that n and n_0 are the quark numbers of $i=1$ and $i=2,3$ in $|m_1\rangle$, respectively. The eigenvalue equations for $\hat{n}$ and $\hat{n}_0$ give these numbers. Since $\hat{n}-\hat{n}_0 = \hat{a}^*\hat{a}+\sum_i \hat{b}_i^*\hat{b}_i$ and, then, $(\hat{n}-\hat{n}_0)$ is positive definite, we have

$$n \geq n_0\ . \tag{4·7}$$

The operators $\hat{n}_0$ and $\hat{n}$ commute with the $su(4)$-generators and we have

$$\hat{n}_0|\lambda\rho\sigma_0\sigma\rangle = n_0|\lambda\rho\sigma_0\sigma\rangle\ , \tag{4·8a}$$

$$\hat{n}|\lambda\rho\sigma_0\sigma\rangle = n|\lambda\rho\sigma_0\sigma\rangle\ . \tag{4·8b}$$

The relation (4·8) suggests us that even if the eigenstates of $\hat{H}_m$ are not expressed explicitly in terms of the minimum weight states such as the form (4·4), we can determine the quark numbers n_0 and n charactering the minimum weight states.

As a possible expression of the eigenstates for $\hat{H}_m$, we presented the following form in (B):

$$\|lsrw\rangle = (\hat{S}^3)^{2l}(\hat{q}^1)^{2s}(\hat{B}^*)^{2r}(\hat{b}^*)^{2w}|0\rangle\ . \tag{4·9}$$

Here, $\hat{S}^3$, $\hat{q}^1$ and $\hat{B}^*$ are defined in the relations (3·1), (3·11) and (3·12), respectively. In (B), we proved that the expressions (4·4) and (4·9) are equivalent to each other under the correspondence

$$l=\lambda\ , \quad s=\sigma-\sigma_0-\rho\ , \quad r=\rho\ , \quad w=\sigma\ . \tag{4·10}$$

Since the operators $\hat{q}^i$, $\hat{S}^j$ and $\hat{B}^*$ commute with one another, the ordering of these operators in $\|lsrw\rangle$ is arbitrary. As is shown in the relation (3·15), $\hat{q}^1$, $\hat{S}^3$ and $\hat{B}^*$ in the state $\|lsrw\rangle$ carry one, two and three quarks. This indicates that the state (4·9) is expressed in terms of the product of single-quarks, quark-pairs and quark-triplets. The above-mentioned structure of the state (4·9) is quite interesting, and in this sense, the expression (4·9) seems to be superior to the expression (4·4). In (B), we showed that, in relation to the $su(1,1)$-algebra, the state (4·9) is classified into two groups. First group obeys the condition $0 \leq r \leq (w-(s+l))/2$ and the second obeys $(w-(s+l))/2 \leq r \leq w-(s+l)$. They are shown in the relations (B·5·26) and (B·5·39), respectively. We are interested to investigate how the single-quarks, the quark-pairs and the quark-triplets coexist with one another in many-quark system. Therefore, irrelevantly to the magnitude of s and l, it may be important to investigate the case $r=0$, in which there does not exist a quark-triplet. From the above reason, in this paper, we will investigate the case of the first group.

With the use of the relations (3·16) $\sim$ (3·18), we can prove the following relations:

$$\hat{S}_2^1\|lsrw\rangle = \hat{S}_3^1\|lsrw\rangle = \hat{S}_3^2\|lsrw\rangle = 0\ , \tag{4·11a}$$

$$\begin{aligned}
&\hat{S}_1^1\|lsrw\rangle = 2(l+2r-w)\|lsrw\rangle\ , \\
&\hat{S}_2^2\|lsrw\rangle = 2(l+s+2r-w)\|lsrw\rangle\ , \\
&\hat{S}_3^3\|lsrw\rangle = 2(2l+s+2r-w)\|lsrw\rangle\ .
\end{aligned} \tag{4·11b}$$

The state (4·9) is the eigenstate of $\hat{n}_0$, $\hat{n}$, $\hat{N}$ and $\hat{\boldsymbol{Q}}^2$:

$$\hat{n}_0\|lsrw\rangle = n_0\|lsrw\rangle\ , \qquad n_0 = \Omega - w\ , \tag{4·11c}$$
$$\hat{n}\|lsrw\rangle = n\|lsrw\rangle\ , \qquad n = \Omega - w + 2(s+r)\ , \tag{4·11d}$$
$$\hat{N}\|lsrw\rangle = N\|lsrw\rangle\ , \qquad N = 3(\Omega - w) + 2(s+2l+3r)\ , \tag{4·11e}$$
$$\hat{\boldsymbol{Q}}^2\|lsrw\rangle = Q^2\|lsrw\rangle\ , \qquad Q^2 = \frac{8}{3}(s^2+sl+l^2) + 4(s+l)\ . \tag{4·11f}$$

With the use of the relation (4·10), the eigenvalues of $\hat{n}_0$ and $\hat{n}$ become the forms shown in the relations (4·5a) and (4·5b), respectively. It may be interesting to see that the state (4·9) is not expressed with the explicit use of $|m_1\rangle$, but, with the aid of $\hat{n}_0$ and $\hat{n}$, we can determine the eigenvalues of $\hat{n}_0$ and $\hat{n}$ characterizing $|m_1\rangle$ in the same results as that given in the state (4·4). The relation (4·11) tells us that $\|lsrw\rangle$ is a minimum weight state of the $su(3)$-algebra based on the case (A·4·5). Further, $\|lsrw\rangle$ is also a minimum weight state of the $su(1,1)$-algebra:

$$\hat{T}_-\|lsrw\rangle = 0\ , \qquad \hat{T}_0\|lsrw\rangle = (w+2)\|lsrw\rangle\ . \tag{4·12}$$

With the use of the relations (4·11) and (4·12), we can construct the complete orthogonal set for the present system consisting of the eight kinds of bosons, which is shown in the relation (B·4·8).

The operation of $\hat{S}_1^2$, $\hat{S}_1^3$ and $\hat{S}_2^3$ on $\|lsrw\rangle$ leads us to

$$\begin{aligned}
&\hat{S}_1^2\|lsrw\rangle = -2s\hat{q}^2\|ls-r/2w\rangle\ ,\\
&\hat{S}_1^3\|lsrw\rangle = 2l\hat{S}^1\|l-s/2rw\rangle - 2s\hat{q}^3\|ls-r/2w\rangle\ ,\\
&\hat{S}_2^3\|lsrw\rangle = 2l\hat{S}^2\|l-s/2rw\rangle\ .
\end{aligned} \tag{4·13}$$

Combining the relations (4·11) and (4·13) with the condition (2·9), we can learn that the state $\|lsrw\rangle$ becomes color-singlet only in the case

$$l = s = 0\ , \quad \text{i.e.,} \quad \|l=0\ s=0\ rw\rangle = (\hat{B}^*)^{2r}(\hat{b}^*)^{2w}|0\rangle\ . \tag{4·14}$$

The above indicates that only the states consisting of the quark-triplets are color-singlet. However, even if the condition (2·11) is applied to $\|lsrw\rangle$, the above conclusion does not change. Certainly, for any values of l and s, we have

$$\langle lsrw\|\hat{S}_i^j\|lsrw\rangle = 0 \quad \text{for} \quad i \neq j\ . \tag{4·15}$$

However, for the expectation values of $\hat{S}_i^i$ for the normalized $\|lsrw\rangle$, we have

$$\begin{aligned}
&\langle lsrw\|\hat{S}_1^1\|lsrw\rangle = 2(l+2r-w)\ ,\\
&\langle lsrw\|\hat{S}_2^2\|lsrw\rangle = 2(l+s+2r-w)\ ,\\
&\langle lsrw\|\hat{S}_3^3\|lsrw\rangle = 2(2l+s+2r-w)\ .
\end{aligned} \tag{4·16}$$

The relation (4·16) tells us that except the case $l = s = 0$, the expectation values are not equal. The above conclusion seems to suggest us that the color-singlet states cannot be described in terms of the states constructed on a chosen single minimum weight state, for example, such as $|m_1\rangle$. Of course, the case (4·14) is an exception, i.e., the case $l = s = 0$. If the above conclusion is reasonable, our next task is to investigate other types of the minimum weight states.

§5. Construction of the "color-singlet" states based on the minimum weight states under the permutation for the color quantum numbers

In this section, we will investigate the state $|cs\rangle_L$, the counterpart of $|cs)_L$. As was shown in the relations (2·4) and (2·5), there exist six cases for fixing the $su(3)$-generators in the $su(4)$-algebra. The minimum weight state which we have adopted until the present is the state $|m_1\rangle$ given in the form (4·3). It comes from one of the six cases, i.e., the case (2·4a). In order to approach our problem for obtaining the "color-singlet" states, it may be indispensable to examine other five cases.

The five minimum weight states are derived formally by the permutations for $i = 1,\ 2$ and 3 from $|m_1\rangle$, and then, for a moment, we denote $|m_1\rangle$ as $|m_{123}\rangle$. Under this notation, the six minimum weight states can be expressed as $|m_{i_1 i_2 i_3}\rangle$. Here, $(i_1 i_2 i_3)$ is obtained from (123) by the permutation $(1 \to i_1, 2 \to i_2, 3 \to i_3)$. The case $(i_1 = 1, i_2 = 2, i_3 = 3)$ is identical permutation. Under the above notation, the minimum weight states coming from the cases (2·4a) $\sim$ (2·4c) are expressed as

$$|m_{123}\rangle = (\hat{b}_1^*)^{2(\sigma-\sigma_0)}(\hat{b}^*)^{2\sigma}|0\rangle \ (= |m_1\rangle) \ , \tag{5·1a}$$
$$|m_{231}\rangle = (\hat{b}_2^*)^{2(\sigma-\sigma_0)}(\hat{b}^*)^{2\sigma}|0\rangle \ (= |m_2\rangle) \ , \tag{5·1b}$$
$$|m_{312}\rangle = (\hat{b}_3^*)^{2(\sigma-\sigma_0)}(\hat{b}^*)^{2\sigma}|0\rangle \ (= |m_3\rangle) \ . \tag{5·1c}$$

The cases (2·5a) $\sim$ (2·5c) give us

$$|m_{132}\rangle = |m_1\rangle \ , \qquad |m_{213}\rangle = |m_2\rangle \ , \qquad |m_{321}\rangle = |m_3\rangle \ . \tag{5·2}$$

The eigenstates of $\widetilde{H}_m$ are obtained by operating the $su(4)$-generators appropriately on the minimum weight states. Therefore, it may be enough to consider the states $|m_1\rangle$, $|m_2\rangle$ and $|m_3\rangle$. Of course, $|m_1\rangle$, $|m_2\rangle$ and $|m_3\rangle$ satisfy

$$\begin{aligned}
&\hat{S}_1|m_1\rangle = \hat{S}_2|m_1\rangle = \hat{S}_3|m_1\rangle = 0 \ , \\
&\hat{S}_2^1|m_1\rangle = \hat{S}_3^1|m_1\rangle = \hat{S}_3^2|m_1\rangle = 0 \ , \\
&\hat{S}_1^1|m_1\rangle = -2\sigma|m_1\rangle \ , \qquad \hat{S}_2^2|m_1\rangle = \hat{S}_3^3|m_1\rangle = -2\sigma_0|m_1\rangle \ ,
\end{aligned} \tag{5·3a}$$

$$\begin{aligned}
&\hat{S}_1|m_2\rangle = \hat{S}_2|m_2\rangle = \hat{S}_3|m_2\rangle = 0 \ , \\
&\hat{S}_3^2|m_2\rangle = \hat{S}_1^2|m_2\rangle = \hat{S}_1^3|m_2\rangle = 0 \ , \\
&\hat{S}_2^2|m_2\rangle = -2\sigma|m_2\rangle \ , \qquad \hat{S}_3^3|m_2\rangle = \hat{S}_1^1|m_2\rangle = -2\sigma_0|m_2\rangle \ ,
\end{aligned} \tag{5·3b}$$

$$\begin{aligned}
&\hat{S}_1|m_3\rangle = \hat{S}_2|m_3\rangle = \hat{S}_3|m_3\rangle = 0 \ , \\
&\hat{S}_1^3|m_3\rangle = \hat{S}_2^3|m_3\rangle = \hat{S}_2^1|m_3\rangle = 0 \ , \\
&\hat{S}_3^3|m_3\rangle = -2\sigma|m_3\rangle \ , \qquad \hat{S}_1^1|m_3\rangle = \hat{S}_2^2|m_3\rangle = -2\sigma_0|m_3\rangle \ .
\end{aligned} \tag{5·3c}$$

We will construct the orthogonal sets based on the minimum weight states (5·1a) $\sim$ (5·1c).

First, we present the minimum weight states for the $su(3)$-algebra which are orthogonal to $\|lsrw\rangle$ shown in the form (4·9). The state $\|lsrw\rangle$ is constructed on the

state $|m_1\rangle$ $(= |m_{123}\rangle)$. Hereafter, we express the state $\|lsrw\rangle$ in the following form:

$$\|123; slrw\rangle = N_{slrw}(\hat{q}^1)^{2s}(\hat{S}^3)^{2l}(\hat{B}^*)^{2r}(\hat{b}^*)^{2w}|0\rangle \quad (= \|1; slrw\rangle) \ . \tag{5·4a}$$

With the aim of using the cyclic permutation $(1 \to 2, 2 \to 3, 3 \to 1)$, we changed the notation $\|lsrw\rangle$ to $\|1; slrw\rangle$, where s and l are transposed and N_{slrw} denotes the normalization constant. By the cyclic permutation, we obtain other two states constructed on $|m_2\rangle$ $(= |m_{231}\rangle)$ and $|m_3\rangle$ $(= |m_{312}\rangle)$ in the form

$$\|231; slrw\rangle = N_{slrw}(\hat{q}^2)^{2s}(\hat{S}^1)^{2l}(\hat{B}^*)^{2r}(\hat{b}^*)^{2w}|0\rangle \quad (= \|2; slrw\rangle) \ , \tag{5·4b}$$

$$\|312; slrw\rangle = N_{slrw}(\hat{q}^3)^{2s}(\hat{S}^2)^{2l}(\hat{B}^*)^{2r}(\hat{b}^*)^{2w}|0\rangle \quad (= \|3; slrw\rangle) \ . \tag{5·4c}$$

In the case $s = l = 0$, the above three states become identical: $\|i; s = 0\ l = 0\ rw\rangle = N_{s=0l=0rw}(\hat{B}^*)^{2r}(\hat{b}^*)^{2w}|0\rangle$. It is color-singlet state $|cs\rangle$. Therefore, hereafter, we consider the cases except the case $s = l = 0$. The above three states are the minimum weight states of the $su(3)$-algebra:

$$\hat{S}_2^1\|1; slrw\rangle = \hat{S}_3^1\|1; slrw\rangle = \hat{S}_3^2\|1; slrw\rangle = 0 \ , \tag{5·5a}$$

$$\hat{S}_3^2\|2; slrw\rangle = \hat{S}_1^2\|2; slrw\rangle = \hat{S}_1^3\|2; slrw\rangle = 0 \ , \tag{5·5b}$$

$$\hat{S}_1^3\|3; slrw\rangle = \hat{S}_2^3\|3; slrw\rangle = \hat{S}_2^1\|3; slrw\rangle = 0 \ , \tag{5·5c}$$

$$\begin{aligned}
\hat{S}_1^1\|1; slrw\rangle &= 2(l + 2r - w)\|1; slrw\rangle \ , \\
\hat{S}_1^1\|2; slrw\rangle &= 2(s + 2l + 2r - w)\|2; slrw\rangle \ , \\
\hat{S}_1^1\|3; slrw\rangle &= 2(s + l + 2r - w)\|3; slrw\rangle \ ,
\end{aligned} \tag{5·6}$$

$$\begin{aligned}
\hat{S}_2^2\|1; slrw\rangle &= 2(s + l + 2r - w)\|1; slrw\rangle \ , \\
\hat{S}_2^2\|2; slrw\rangle &= 2(l + 2r - w)\|2; slrw\rangle \ , \\
\hat{S}_2^2\|3; slrw\rangle &= 2(s + 2l + 2r - w)\|3; slrw\rangle \ ,
\end{aligned} \tag{5·7}$$

$$\begin{aligned}
\hat{S}_3^3\|1; slrw\rangle &= 2(s + 2l + 2r - w)\|1; slrw\rangle \ , \\
\hat{S}_3^3\|2; slrw\rangle &= 2(s + l + 2r - w)\|2; slrw\rangle \ , \\
\hat{S}_3^3\|3; slrw\rangle &= 2(l + 2r - w)\|3; slrw\rangle \ .
\end{aligned} \tag{5·8}$$

The relation (5·6) $\sim$ (5·8) tell us that the states are orthogonal to one another. With the use of the relations (5·5a) $\sim$ (5·5c), we can prove that all the matrix elements except the following ones vanish:

$$\begin{aligned}
&\langle 2; s'l'r'w'\|\hat{S}_1^i\|1; slrw\rangle \quad \text{for} \quad i = 2,\ 3 \ , \\
&\langle 3; s'l'r'w'\|\hat{S}_2^i\|2; slrw\rangle \quad \text{for} \quad i = 3,\ 1 \ , \\
&\langle 1; s'l'r'w'\|\hat{S}_3^i\|3; slrw\rangle \quad \text{for} \quad i = 1,\ 2 \ .
\end{aligned} \tag{5·9}$$

However, these matrix elements also vanish. For example, $\hat{S}_1^i \| 1; slrw\rangle$ is not constructed by operating $\hat{S}_1^i$ on the state $\|2; slrw\rangle$ and this indicates that it does not belong to the irreducible representation based on the minimum weight state $\|2; slrw\rangle$. From the above argument, the matrix element $\langle 2; s'l'r'w' \| \hat{S}_1^i \| 1; slrw\rangle$ vanishes. Other cases are also in the same situation as the above one. The above minimum weight states are the eigenstates of $\hat{n}_0$ and $\hat{n}$ with the eigenvalues $n_0 = \Omega - w = \Omega - \sigma$ and $n = \Omega - w + 2(s+r) = \Omega + \sigma - 2\sigma_0$, respectively. These values are shown in the relations (4·11c) and (4·11d). The reason is very simple: The operators $\hat{n}_0$ and $\hat{n}$ are color-symmetric.

Under the above consideration, we will construct the "color-singlet" states in the present model. In the case except the case $s = l = 0$, it is impossible to construct the states obeying the condition (2·9). Then, we take up the condition (2·11). First, we set up a state which is a linear combination for the states $\|p; slrw\rangle$ for $p = 1,\ 2$ and 3:

$$\|cs; slrw\rangle = \sum_p C_p(slrw)\|p; slrw\rangle \ . \tag{5·10}$$

Here, $C_p(slrw)$ denotes the coefficient of the linear combination with $\sum_p |C_p(slrw)|^2 = 1$. We note that for any values of $C_p(slrw)$, the state (5·10) satisfies

$$\langle cs; slrw \| \hat{S}_i^j \| cs; slrw\rangle = 0 \quad \text{for} \quad i \neq j \ . \tag{5·11a}$$

The relation (5·11a) is nothing but the condition (2·11a). Concerning the condition (2·11b), we are able to have

$$\begin{aligned} &\langle cs; slrw \| \hat{S}_1^1 \| cs; slrw\rangle = \langle cs; slrw \| \hat{S}_2^2 \| cs; slrw\rangle \\ &= \langle cs; slrw \| \hat{S}_3^3 \| cs; slrw\rangle = \frac{4}{3}(s+2l) + 2(2r-w) \ . \end{aligned} \tag{5·11b}$$

The relation (5·11b) is realized in the case

$$|C_p(slrw)|^2 = \frac{1}{3} \ . \tag{5·12}$$

Any case except the relation (5·12) does not lead to the condition (2·11b). In the present framework, it is impossible to fix the phase factor of $C_p(slrw)$, but, it may be enough to adopt the following form:

$$\begin{aligned} |cs\rangle_L = \|cs; slrw\rangle &= \frac{1}{\sqrt{3}} \sum_p \|p; slrw\rangle \\ &= \frac{1}{\sqrt{3}}(\|123; slrw\rangle + \|231; slrw\rangle + \|312; slrw\rangle) \ . \end{aligned} \tag{5·13}$$

It may be permitted to say that the state $|cs\rangle_L = \|cs; slrw\rangle$ is color-symmetric state with respect to the cyclic permutation for $p = 1, 2$ and 3. Since $\hat{n}_0$, $\hat{n}$ and $\hat{N}$ are color-symmetric, the results obtained in the state $\|lsrw\rangle$ are also found in the state $\|cs; slrw\rangle$. The results are shown in the relations (4·11c) $\sim$ (4·11f). The energy

eigenvalue is also in the same situation as the above. The details will be discussed in the next section. The above argument supports that any information related to the eigenvalue of color-symmetric operator given in (A) $\sim$ (C) is also valid in the "color-singlet" state $|cs\rangle_L$.

At the end of §2, we mentioned that it may be desirable to search the state $|cs)_L$, in which $(cs|\widetilde{\boldsymbol{Q}}^2|cs)_L$ is as small as possible, i.e., $|cs)_M$. We complement this statement. There does not exist the absolute condition for the state $|cs)_M$ which we intend to search. In the next section, we will give a possible idea for this condition.

§6. Construction of the "color-singlet" states minimizing the eigenvalue of the $su(3)$-Casimir operator

In this section, we will investigate the state $|cs\rangle_M$, the counterpart of $|cs)_M$. In the relation (5·13), we presented the "color-singlet" states $|cs\rangle_L$ which obey the condition (2·11). After discussing some characteristic features produced by these states, we will formulate the states $|cs\rangle_M$ in our idea. As was already mentioned in §4, the expressions (4·4) and (4·9) are equivalent to each other through the relation (4·10) for the quantum numbers specifying the states. The relation (4·10) is also valid for the case of the states $|cs\rangle_L$. In (A), after lengthy argument, we showed the overall ranges in which the quantum numbers can change. In this paper, we investigate the range based on the first group mentioned in §4.

First, we notice the following relation:

$$2s \geq 0 \ , \qquad 2l \geq 0 \ , \qquad 2r \geq 0 \ , \qquad 2w \geq 0 \ , \tag{6·1a}$$

$$2s + 2l + 4r \leq 2w \ . \tag{6·1b}$$

The relation (6·1a) comes from the condition that the exponents $2s$, $2l$, $2r$ and $2w$ in the relation (5·4a) should be positive or zero. Strictly speaking, the quantum numbers $2s$, $2l$, $2r$ and $2w$ are positive integers. However, for simplicity, we will treat them as continuously varying positive parameters. The relation (6·1b) comes from the first group for $\|cs; slrw\rangle$. It should be noted that $2s$, $2l$ and $2r$ denote the numbers of the single-quarks, the quark-pairs and the quark-triplets, respectively. Further, we note the eigenvalues of the eigenvalue equations (4·11c)$\sim$(4·11e), which lead to

$$w = \Omega^0 \ , \tag{6·2a}$$

$$2s + 2r = n^0 \ , \qquad 2l + 2r = \frac{1}{2}(N^0 - n^0) \ . \tag{6·2b}$$

The relations (6·2a) and (6·2b) give us

$$(2s + 2l + 4r) - 2w = \frac{1}{2}(N^0 + n^0) - 2\Omega^0 \ . \tag{6·2c}$$

Here, Ω^0, n^0 and N^0 are defined as

$$\Omega^0 = \Omega - n_0 \ , \qquad n^0 = n - n_0 \ , \qquad N^0 = N - n_0 \ . \tag{6·3}$$

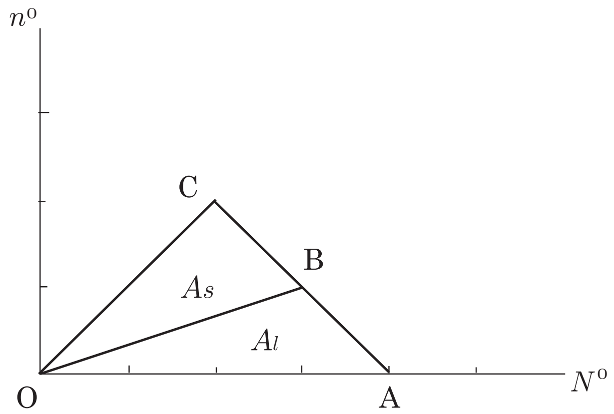

Fig. 1. Two areas in which n^0 and N^0 satisfy for a given value Ω^0 are depicted. The line OA represents $n^0 = 0$, the line OB represents $n^0 = N^0/3$, the line OC represents $n^0 = N^0$ and the line AC represents $n^0 = 4\Omega^0 - N^0$.

The quantities Ω^0, n^0 and N^0 are also regarded as continuously varying positive parameters. Combining the relation (6·2) with the inequality (6·1), we have the inequality.

$$0 \le n^0 \ , \qquad n^0 \le N^0 \ , \qquad n^0 \le 4\Omega^0 - N^0 \ . \tag{6·4}$$

The relation (6·2b) gives us

$$2s - 2l = \frac{3}{2}\left(n^0 - \frac{N^0}{3}\right) \ . \tag{6·5}$$

From the relation (6·5), we have the following relation:

$$\text{if} \qquad 2s \le 2l \ , \qquad n^0 \le \frac{N^0}{3} \ , \tag{6·6a}$$

$$\text{if} \qquad 2s \ge 2l \ , \qquad n^0 \ge \frac{N^0}{3} \ . \tag{6·6b}$$

Figure 1 shows two areas A_l and A_s, where the inequalities (6·4) and (6·6) are satisfied. The line OA gives $2s = 2r = 0$ and $2l = N^0/2$, in which the system consists of only $N^0/2$ quark-pairs. On the line OC, we have $2l = 2r = 0$ and $2s = N^0$. In this case, the system consists of only N^0 single-quarks. On the line OB, we have $2s = 2l = N^0/3 - 2r$. In this case, the numbers of the single-quarks and the quark-pairs are equal to each other. If $2s = 2l = 0$, $2r = N^0/3$ and this case corresponds to the quark-triplets and, of course, if $2r = 0$, we have $2s = 2l = N^0/3$. In the area A_l, the number of the quark-pairs is larger than that of the single-quarks and in the area A_s, the vice versa. The relation (6·2b) gives us $2s = n^0 - 2r \ge 0$ and $2l = (N^0 - n^0)/2 - 2r \ge 0$ and, then, we can show that in the areas A_l and A_s, $2r$ can changes its value in the following ranges, respectively:

$$A_l \ : \quad 0 \le 2r \le n^0 \ , \tag{6·7a}$$

$$A_s \ : \quad 0 \le 2r \le \frac{1}{2}(N^0 - n^0) \ . \tag{6·7b}$$

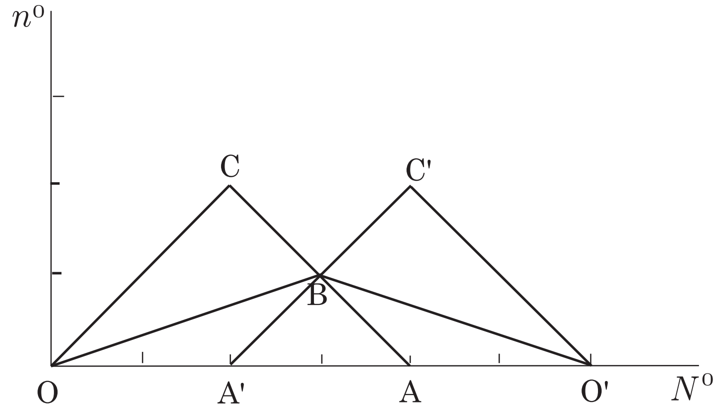

Fig. 2. New triangle A'O'C' is obtained from old triangle AOC by replacing N^0 with $(6\Omega - N^0)$.

Later, the relation (6·7) will play a central role.

In Fig.1, we see that N^0 changes its value in the range

$$0 \leq N^0 \leq 4\Omega^0 \ . \tag{6·8}$$

However, the range of N in the present model is as follows:

$$3n^0 \leq N \leq 6\Omega - 3n^0 \ , \quad \text{i.e.,} \quad 0 \leq N^0 \leq 6\Omega^0 \ . \tag{6·9}$$

Therefore, we must interpret the discrepancy between $4\Omega^0$ and $6\Omega^0$. As was discussed in §2.3 in (A), we can make re-formation of the present model from the side of $N = 6\Omega$, which we called the hole picture in (A). If we follow this re-formation, it is enough to replace N^0 with $(6\Omega^0 - N^0)$ in the relation obtained as function of N^0 in the present form:

$$2\Omega^0 \leq N^0 \leq 6\Omega^0 \ , \quad 0 \leq n^0 \ , \quad n^0 \leq 6\Omega^0 - N^0 \ , \quad n^0 \leq -2\Omega^0 + N^0 \ . \tag{6·10}$$

The line $n^0 = N^0/3$ changes to $n^0 = 2\Omega^0 - N^0/3$. Figure 2 shows the areas given in the relation (6·10), together with the areas given in the relation (6·4). We can see that both forms cover the whole ranges.

The eigenvalue of the $su(3)$-Casimir operator, Q^2, is shown in the relation (4·11f). It is expressed in terms of $2s$ and $2l$ and using the relation (6·2b), it can be expressed in terms of N^0, n^0 and $2r$. Hereafter, we denote Q^2 as $F(N^0, n^0; 2r)$:

$$F(N^0, n^0; 2r) = 2\left[2r - \left(\frac{1}{4}(N^0 + n^0) + 1\right)\right]^2 + \frac{3}{8}\left[\left(n^0 - \frac{N^0}{3}\right)^2 - \frac{16}{3}\right] . \tag{6·11}$$

Since Q^2 is positive-definite, for given values of N^0 and n^0, $2r$ is meaningful in the regions satisfying the condition $F(N^0, n^0; 2r) \geq 0$. For given values of N^0 and n^0, they appear in the regions

(1) If $n^0 \geq \dfrac{N^0}{3} + \dfrac{4}{\sqrt{3}}$ or $n^0 \leq \dfrac{N^0}{3} - \dfrac{4}{\sqrt{3}}$,

$$-\infty < 2r < +\infty \ , \tag{6·12a}$$

(2) If $\dfrac{N^0}{3} - \dfrac{4}{\sqrt{3}} \leq n^0 \leq \dfrac{N^0}{3} + \dfrac{4}{\sqrt{3}}$,

$$2r \leq \frac{1}{4}(N^0 + n^0) + 1 - \sqrt{1 - \frac{3}{16}\left(n^0 - \frac{N^0}{3}\right)^2} \ , \tag{6·12b}$$

$$2r \geq \frac{1}{4}(N^0 + n^0) + 1 + \sqrt{1 - \frac{3}{16}\left(n^0 - \frac{N^0}{3}\right)^2} \ . \tag{6·12c}$$

In the case (1), we can prove

$$n^0 \leq \frac{1}{4}(N^0 + n^0) + 1 \qquad \text{for} \quad n^0 \leq \frac{N^0}{3} - \frac{4}{\sqrt{3}} \ , \tag{6·13a}$$

$$\frac{1}{2}(N^0 - n^0) \leq \frac{1}{4}(N^0 + n^0) + 1 \quad \text{for} \quad n^0 \geq \frac{N^0}{3} + \frac{4}{\sqrt{3}} \ . \tag{6·13b}$$

In the case (2), we can prove

$$n^0 \leq \frac{1}{4}(N^0 + n^0) + 1 - \sqrt{1 - \frac{3}{16}\left(n^0 - \frac{N^0}{3}\right)^2}$$
$$\text{for} \quad \frac{N^0}{3} - \frac{4}{\sqrt{3}} \leq n^0 \leq \frac{N^0}{3} \ , \tag{6·14a}$$

$$\frac{1}{2}(N^0 - n^0) \leq \frac{1}{4}(N^0 + n^0) + 1 - \sqrt{1 - \frac{3}{16}\left(n^0 - \frac{N^0}{3}\right)^2}$$
$$\text{for} \quad \frac{N^0}{3} \leq n^0 \leq \frac{N^0}{3} + \frac{4}{\sqrt{3}} \ . \tag{6·14b}$$

The function $F(N^0, n^0; 2r)$ is monotone-decreasing in the range $0 \leq 2r \leq (N^0 + n^0)/4 + 1$. Therefore, the maximum values of $2r$, $2r_m$, are given as follows:

$$2r_m = n^0 \qquad \text{in A}_l \ , \tag{6·15a}$$

$$2r_m = \frac{1}{2}(N^0 - n^0) \quad \text{in A}_s \ . \tag{6·15b}$$

The behavior of $F(N^0, n^0; 2r)$ leads to the following results: The minimum values of $F(N^0, n^0; 2r)$ are given by $F(N^0, n^0; 2r_m)$ for given values of N^0 and n^0 and their values are explicitly calculated in the forms

$$F(N^0, n^0, 2r_m = n^0) = \frac{1}{6}(N^0 - 3n^0)^2 + (N^0 - 3n^0) \qquad \text{for A}_l \ , \tag{6·16a}$$

$$F(N^0, n^0, 2r_m = \frac{1}{2}(N^0 - n^0)) = \frac{1}{6}(N^0 - 3n^0)^2 - (N^0 - 3n^0) \quad \text{for A}_s \ . \tag{6·16b}$$

In the forthcoming paper (Part II), we use the notations for $F(N^0, n^0, 2r_m)$ and the "color-singlet" states $|cs\rangle_M$ leading to $F(N^0, n^0; 2r_m)$ in the form $F_M(N^0, n^0)$ and

$|cs; N^0, n^0\rangle_M$, respectively. From the above argument, it may be natural to regard $|cs; N^0, n^0\rangle_M$ as the "color-singlet" states $|cs\rangle_M$ which we have intended to search in this paper, because the states $|cs; N^0, n^0\rangle_M$ minimize $\langle cs|\hat{\boldsymbol{Q}}^2|cs\rangle_L$ in the areas A_l and A_s. It satisfies our requirement mentioned in §2. Of course, we can give the explicit expression for $|cs; N^0, n^0\rangle_M$, but it may be not necessary for present argument.

In this paper, we presented the color-symmetric form of the $su(4)$-algebraic model for many-quark system. With the aid of the Schwinger boson realization, this form automatically leads to the "color-singlet" states in the boson space, which satisfy the relation equivalent to the relation (2·11) in the original fermion space. The color-symmetric hermitian operators, $\hat{\boldsymbol{P}}^2$, $\hat{\boldsymbol{Q}}^2$, $\hat{N}$ and $\hat{n}$ are essential for the form. In various effective theories of QCD, the condition such as the relation (2·11) plays a central role. However, in this paper, we pointed out that it may be sufficient to insure the color-singlet property, only in the frame of the condition (2·11). Further, we stressed that, in addition to the condition (2·11), the expectation value of $\hat{\boldsymbol{Q}}^2$ should be as small as possible. In the forthcoming paper (Part II), we will give various analysis for this condition.

In order to complete the present paper (Part I), we will give the expression of the energy eigenvalues. With the use of the expression (2·8b), we can calculate the energy eigenvalues of $\hat{H}_m$ for the states $\|cs; slrw\rangle$. We know that the eigenvalues of $\hat{\boldsymbol{Q}}^2$ and $\hat{\Sigma}$ are expressed in the forms $F(N^0, n^0; 2r)$ and $(2\Omega^0 - N^0)(3\Omega^0 - N^0 + 6)/3$, respectively. The relations (A·2·2b) and (4·5) give us the eigenvalue of $\hat{\boldsymbol{P}}^2$ in the form $(3\Omega^{02} - 2\Omega^0 n^0 + n^{02} + 6\Omega^0)$. Then, the energy eigenvalue of $\hat{H}_m$, which we denote as $E^{(m)}(N^0, n^0; 2r)$, is obtained as follows:

$$\begin{aligned}E^{(m)}(N^0, n^0; 2r) &= \frac{1}{2}(1 + 2\chi)F(N^0, n^0; 2r) \\ &\quad + \frac{1}{2}n^0(2\Omega^0 - n^0) - \frac{1}{6}N^0(6\Omega^0 + 6 - N^0) \ . \qquad (6\cdot17)\end{aligned}$$

Of course, the expression (6·17) is given for the case $0 \le N^0 \le 4\Omega^0$. For the case $2\Omega^0 \le N^0 \le 6\Omega^0$, we have the expression for the energy eigenvalue, which we denote as $\mathcal{E}^{(m)}(N^0, n^0; 2r)$, in the form

$$\mathcal{E}^{(m)}(N^0, n^0; 2r) = E^{(m)}(6\Omega^0 - N^0, n^0; 2r) + 2(3\Omega^0 - N^0) \ . \qquad (6\cdot18)$$

The above expression is obtained by replacing N^0 with $(6\Omega^0 - N^0)$ in $E^{(m)}(N^0, n^0; 2r)$ and by adding the term $2(3\Omega^0 - N^0)$, which comes from the difference of the ordering of $\hat{S}^i\hat{S}_i$ and $\hat{S}_i\hat{S}^i$. The detail has been discussed in (A). Hereafter, we will not contact with the case $2\Omega^0 \le N^0 \le 6\Omega^0$ explicitly. Of course, in the case where we discuss some aspects of the present model, we will use the results for $2\Omega^0 \le N^0 \le 6\Omega^0$, if necessary.

By substituting the relation (6·16) into the expression (6·17), we obtain the energy eigenvalue for the state $|cs; N^0, n^0\rangle_M$:

For the area A_l ,

$$E_l^{(m)}(N^0, n^0) = E^{(m)}(N^0, n^0; 2r_m)$$

$$= \frac{1}{4}(1+6\chi)n^{02} - \frac{1}{2}\left[(N^0+3-2\Omega^0)+2\chi(N^0+3)\right]n^0$$
$$-\frac{1}{4}N^0(4\Omega^0-N^0+2)+\frac{\chi}{6}N^0(N^0+6)\ , \qquad (6\cdot 19\mathrm{a})$$

For the area A_s ,

$$E_s^{(m)}(N^0,n^0) = E^{(m)}(N^0,n^0;2r_m)$$
$$= \frac{1}{4}(1+6\chi)n^{02} - \frac{1}{2}\left[(N^0-3-2\Omega^0)+2\chi(N^0-3)\right]n^0$$
$$-\frac{1}{4}N^0(4\Omega^0-N^0+6)+\frac{\chi}{6}N^0(N^0-6)\ . \qquad (6\cdot 19\mathrm{b})$$

In the discussion in Part II, the expressions (6·17) and (6·18) will play a central role. Through the discussion, we may be able to understand how the condition required for the expectation value of $\hat{\boldsymbol{Q}}^2$ will work in the energies.

§7. Summary

In this paper, we investigated the "color-singlet" state in the modified Bonn quark model by means of the boson realization. In the exact energy eigenstates, a color-neutral quark-triplet state is, of course, a color-singlet state under the ordinary color-singletness. However, we loosened this condition in terms of expectation values. In addition to the above-mentioned condition, we imposed a condition that the "color-singlet" state minimizes the eigenvalue of the $su(3)$-Casimir operator. Then, we could construct the state as the superposition of the eigenstates constructed on the minimum weight states of some $su(3)$-subalgebras included in the dynamical $su(4)$-symmetry which the original Bonn quark model has. Further, we analyzed the regions which consist of the single-quarks, the quark-pairs and the quark-triplets. The implication of the results on the color superconducting phase or nuclear matter phase in dense quark or nuclear matter will be investigated as one of future problems.

Appendix A
—— *The $su(3)$-Algebra* ——

Usually, the generators of the $su(3)$-algebra are given as the following eight operators:

$$\widetilde{I}_{\pm,0}\ , \quad \widetilde{M}\ , \quad \widetilde{D}^*_{\pm,}\ , \quad \widetilde{D}_{\pm}\ . \qquad (\mathrm{A}\cdot 1)$$

They obey the commutation relations

$$[\ \widetilde{I}_+\ ,\ \widetilde{I}_-\ ] = 2\widetilde{I}_0\ , \qquad [\ \widetilde{I}_0\ ,\ \widetilde{I}_\pm\ ] = \pm\widetilde{I}_\pm\ , \qquad (\mathrm{A}\cdot 2\mathrm{a})$$
$$[\ \widetilde{I}_{\pm,0}\ ,\ \widetilde{M}_0\ ] = 0\ , \qquad (\mathrm{A}\cdot 2\mathrm{b})$$
$$[\ \widetilde{I}_\pm\ ,\ \widetilde{D}^*_\pm\ ] = 0\ , \qquad [\ \widetilde{I}_\pm\ ,\ \widetilde{D}^*_\mp\ ] = \widetilde{D}^*_\pm\ , \qquad [\ \widetilde{I}_0\ ,\ \widetilde{D}^*_\pm\ ] = \pm\frac{1}{2}\widetilde{D}^*_\pm\ , \qquad (\mathrm{A}\cdot 2\mathrm{c})$$
$$[\ \widetilde{M}_0\ ,\ \widetilde{D}^*_\pm\ ] = \frac{3}{2}\widetilde{D}^*_\pm\ , \qquad (\mathrm{A}\cdot 2\mathrm{d})$$
$$[\ \widetilde{D}^*_+\ ,\ \widetilde{D}^*_-\ ] = 0\ , \qquad [\ \widetilde{D}^*_\pm\ ,\ \widetilde{D}_\pm\ ] = \widetilde{M}_0 \pm \widetilde{I}_0\ , \qquad [\ \widetilde{D}^*_\pm\ ,\ \widetilde{D}^*_\mp\ ] = \widetilde{I}_\pm\ . \qquad (\mathrm{A}\cdot 2\mathrm{e})$$

The relation (A·2a) tells that $(\widetilde{I}_{\pm,0})$ forms the $su(2)$-algebra. In the relations (A·2b) and (A·2c), we learn that $\widetilde{M}_0$ is scalar and $\widetilde{D}^*_\pm$ and $\widetilde{D}_\mp$ are spinor for $(\widetilde{I}_{\pm,0})$.

In the case (2·4a), the eight $su(3)$-generators can be expressed in the form

$$\widetilde{I}_+ = \widetilde{S}_2^3 \ , \qquad \widetilde{I}_- = \widetilde{S}_3^2 \ , \qquad \widetilde{I}_0 = \frac{1}{2}(\widetilde{S}_2^2 - \widetilde{S}_3^3) \ , \tag{A·3a}$$

$$\widetilde{M}_0 = \widetilde{S}_1^1 - \frac{1}{2}(\widetilde{S}_2^2 + \widetilde{S}_3^3) \ , \tag{A·3b}$$

$$\widetilde{D}^*_+ = \widetilde{S}_1^3 \ , \qquad \widetilde{D}^*_- = \widetilde{S}_1^2 \ , \qquad \widetilde{D}_+ = \widetilde{S}_3^1 \ , \qquad \widetilde{D}_- = \widetilde{S}_2^1 \ . \tag{A·3c}$$

The other cases are obtained under the permutation $(1 \to i_1, 2 \to i_2, 3 \to i_3)$.

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